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FAST TRACK COMMUNICATION

Universal amplitude ratios of two-dimensional percolation from field theory

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Abstract
We complete the determination of the universal amplitude ratios of two-dimensional percolation within the two-kink approximation of the form factor approach. For the cluster size ratio, which has for a long time been elusive both theoretically and numerically, we obtain the value 160.2, in good agreement with the lattice estimate 162.5 ± 2 of Jensen and Ziff.

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Universal combinations of critical amplitudes represent the canonical way of encoding the universal information about the approach to criticality in statistical mechanics [1]. While critical exponents can be determined working *at* criticality, amplitude ratios characterize the scaling region *around* the critical point. They carry independent information about the universality class and their determination is in general theoretically more demanding. Field theory is the natural framework in which to address the problem, but the usual perturbative approach is not helpful if one has to work far below the upper critical dimension d_c.

For the best-known example of a geometric phase transition, namely isotropic percolation ($d_c = 6$) [2], it was shown in [3] how the field theoretical computation of universal amplitude ratios in two dimensions can be addressed non-perturbatively exploiting the fact that percolation can be seen as the $q \to 1$ limit of the q-state Potts model [4], and that the latter is integrable even away from criticality, in the scaling limit for $q \leqslant 4$ [5]. Starting from the exact S-matrix [5] one can compute the Potts correlation functions, and from them the amplitude ratios, using the form factor approach [3].

This programme was completed in [3] for $q = 2, 3, 4$, recovering the known Ising results and obtaining new predictions for the three- and four-state Potts model. For percolation, however, only partial results were obtained, because the determination of some amplitudes above the percolation threshold p_c involves the solution of a functional equation, which in [3] could not be solved for generic values of q, in particular for $q = 1$. In this situation, it was observed in [3] that a simple parabolic extrapolation to $q = 1$ of the results obtained at $q = 2, 3, 4$ produced for the percolation ratios results compatible with the available numerical estimates. In particular, for the ratio of cluster size amplitudes below and above p_c the extrapolated value 74 essentially coincided with the central value of the most recent estimate then available [6].

On the other hand, the status of the numerical results for the cluster size ratio (associated with the Potts susceptibility ratio) was at the time particularly controversial, different authors having obtained over the years values which spanned two orders of magnitude [1]. Following the appearance of [3], Jensen and Ziff communicated a new, very accurate lattice determination of this ratio, essentially coinciding with the value 162.5 ± 2 finally published in [7]. A possible explanation for such a large discrepancy, other than the failure of the extrapolation (which appeared to work for other ratios), was discussed in [8], but was ruled out by the full numerical confirmation the prediction of [3, 8] for the universal ratios at $q = 3$ received[5] in [9, 10] (see also [11]).

In this communication we provide the only piece of analytic information missing in [3], namely the solution of the functional equation at $q = 1$, and show that it leads to results for the percolative universal ratios in complete agreement with the most recent lattice estimates. In particular, this confirms that the only problem with the extrapolated value for the cluster size ratio was the extrapolation itself. When comparing field theoretical and lattice results (table 2) it must be taken into account that, with few exceptions, the former are themselves not exact, since they are obtained truncating the spectral series for correlation functions to the two-kink contribution. The remarkable accuracy of this two-particle approximation is, however, well known (see e.g. the comparison with the Ising exact results in [3]), and is further illustrated by this case.

In [3] the determination in the two-kink approximation of the low-temperature spin-spin correlation function of the scaling q-state Potts model ($q \leqslant 4$) was reduced to that of a function $\Omega_q(\theta)$ entering the two-kink form factor of the spin field. This function is characterized by the following properties [3].

(i) It is a meromorphic function of θ whose only singularity in the strip $\mathrm{Im}\,\theta \in (0, 2\pi)$ is a simple pole at $\theta = \mathrm{i}\pi$ with residue

$$\mathrm{Res}_{\theta=\mathrm{i}\pi}\,\Omega_q(\theta) = \mathrm{i}\frac{q}{q-1}M, \tag{1}$$

where M denotes the Potts spontaneous magnetization.

(ii) It is a solution of the functional equations

$$\Omega_q(\theta) = \Omega_q(-\theta), \tag{2}$$

$$2\cos\frac{\pi\lambda}{3}\sinh(\lambda\theta)\Omega_q(\theta) = \sinh(\lambda(\mathrm{i}\pi+\theta))\Omega_q(2\mathrm{i}\pi-\theta) - \sinh(\lambda(\mathrm{i}\pi-\theta))\Omega_q(2\mathrm{i}\pi+\theta), \tag{3}$$

with the asymptotic behavior

$$\Omega_q(\theta) \sim \exp\left[\left(\frac{2}{3}\lambda - 1\right)\theta\right], \qquad \theta \to +\infty, \tag{4}$$

where λ parameterizes q according to the relation $\sqrt{q} = 2\sin(\pi\lambda/3)$.

For $q \leqslant 3$, where the Potts scattering theory possesses no bound states, the properties (i) and (ii) uniquely identify $\Omega_q(\theta)$, and then the spin field of the scaling Potts model[6].

[5] For the case $q = 4$, which is plagued by logarithmic corrections to scaling [12, 13], the issue of the precise comparison between field theoretical and lattice results for the universal ratios appears still open [8, 9, 14, 15].

[6] See [16, 17] for the correspondence between fields and solutions of the form factor equations in integrable field theory.

In [3] $\Omega_q(\theta)$ was determined only for $q = 2, 3, 4$, where it takes a simple form. We now show that $\Omega_1(\theta)$ can be obtained taking a mathematical detour in the sine-Gordon model. For the latter, which is defined by the action

$$\mathcal{A}_{SG} = \int \mathrm{d}^2x \left(\frac{1}{2}(\partial_\nu\varphi)^2 + \mu\cos\beta\varphi\right), \tag{5}$$

Lukyanov computed in [18] the soliton–antisoliton form factors

$$F^a_{\varepsilon_1\varepsilon_2}(\theta) = \langle 0|\,\mathrm{e}^{\mathrm{i}a\beta\varphi(0)}|A_{\varepsilon_1}(\theta_1)A_{\varepsilon_2}(\theta_2)\rangle, \qquad \varepsilon_i = \pm 1, \qquad \varepsilon_1\varepsilon_2 = -1, \tag{6}$$

obtaining a result that in our notations[7] reads

$$F^a_{\pm\mp}(\theta) = -\langle \mathrm{e}^{\mathrm{i}a\beta\varphi}\rangle \frac{F_0(\theta)}{F_0(\mathrm{i}\pi)} A^a_\pm(\theta), \tag{7}$$

$$A^a_\pm(\theta) = \mathrm{e}^{\mp\frac{\pi}{2\xi}(\mathrm{i}\pi-\theta)}\left[\mathrm{e}^{\mp 2\mathrm{i}\pi a} I_a(-\theta) + I_a(\theta - 2\mathrm{i}\pi)\right], \tag{8}$$

where $\xi = \pi\beta^2/(8\pi - \beta^2)$, $F_0(\theta)$ is a function on which we comment below, and $I_a(\theta)$ is specified *for real values of* θ and $a \in \left(-\frac{1}{2} - \frac{\pi}{\xi}, \frac{1}{2}\right)$ as

$$I_a(\theta) = \mathcal{C}\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{2\pi} W\left(-x - \frac{\theta}{2} + \mathrm{i}\pi\right) W\left(-x + \frac{\theta}{2} + \mathrm{i}\pi\right) \mathrm{e}^{-(\frac{\pi}{\xi}+2a)x}, \tag{9}$$

with

$$W(\theta) = -\frac{2}{\cosh\theta}\exp\left[-2\int_0^\infty \frac{\mathrm{d}t}{t}\frac{\sinh\left(\frac{t}{2}\left(1-\frac{\xi}{\pi}\right)\right)}{\sinh t\,\sinh\frac{\xi t}{2\pi}}\sin^2\left(\frac{t}{2\pi}(\mathrm{i}\pi - \theta)\right)\right], \tag{10}$$

and

$$\mathcal{C} = \frac{1}{4}\exp\left[-4\int_0^\infty \frac{\mathrm{d}t}{t}\frac{\sinh^2\frac{t}{4}\sinh\left(\frac{t}{2}\left(1-\frac{\xi}{\pi}\right)\right)}{\sinh t\,\sinh\frac{\xi t}{2\pi}}\right]. \tag{11}$$

These results were presented in [18] within a framework known as free field representation, which differs from the usual approach to form factors based on functional relations. Of course, this latter approach can be adopted also for the matrix elements (7), using as input the sine-Gordon S-matrix and the fact that the soliton is semi-local with respect to $\mathrm{e}^{\mathrm{i}a\beta\varphi}$, with a semi-locality phase $\mathrm{e}^{2\mathrm{i}\pi a}$. The corresponding functional equations then read [19]

$$F^a_{\varepsilon_1\varepsilon_2}(\theta) = S_T(\theta)F^a_{\varepsilon_2\varepsilon_1}(-\theta) + S_R(\theta)F^a_{\varepsilon_1\varepsilon_2}(-\theta), \tag{12}$$

$$F^a_{\varepsilon_1\varepsilon_2}(\theta + 2\mathrm{i}\pi) = \mathrm{e}^{2\mathrm{i}\pi a\varepsilon_2}F^a_{\varepsilon_2\varepsilon_1}(-\theta), \tag{13}$$

where

$$S_T(\theta) = -\frac{\sinh\frac{\pi\theta}{\xi}}{\sinh\left(\frac{\pi}{\xi}(\theta - \mathrm{i}\pi)\right)}S(\theta), \tag{14}$$

$$S_R(\theta) = -\frac{\sinh\frac{\mathrm{i}\pi^2}{\xi}}{\sinh\left(\frac{\pi}{\xi}(\theta - \mathrm{i}\pi)\right)}S(\theta), \tag{15}$$

are the transmission and reflection amplitudes; the explicit form in the present notations of $S(\theta)$ and $F_0(\theta)$ can be found for example in [20], but here we only need to know that

$$F_0(\theta) = S(\theta)F_0(-\theta), \qquad F_0(\theta + 2\mathrm{i}\pi) = F_0(-\theta). \tag{16}$$

[7] In particular, switching from Lukyanov's notations to ours involves the replacements $\theta \to -\theta$, $\xi \to \xi/\pi$, $a \to \beta a$.

This implies in particular that (13) is automatically satisfied by (7). Since $A^a_\pm$ are meromorphic functions of θ, also I_a, as a linear combination of A^a_+ and A^a_- with entire coefficients is meromorphic. In particular, analyticity implies that the property

$$I_a(\theta) = I_a(-\theta), \tag{17}$$

which for real θ is apparent in (9), extends to the whole complex plane. Requiring (12) leads now to the equation

$$\begin{aligned} 2\cos\left(\frac{\pi^2}{\xi} + 2\pi a\right)\sinh\frac{\pi\theta}{\xi} I_a(\theta) &= \sinh\left(\frac{\pi}{\xi}(\mathrm{i}\pi - \eta\theta)\right) I_a(2\mathrm{i}\pi - \theta) \\ &\quad - \sinh\left(\frac{\pi}{\xi}(\mathrm{i}\pi + \eta\theta)\right) I_a(2\mathrm{i}\pi + \theta), \end{aligned} \tag{18}$$

with $\eta = 1$. Making the identifications

$$\xi = \frac{\pi}{\lambda}, \qquad a = -\frac{\lambda}{2}\left(1 \pm \frac{1}{3}\right) + k, \qquad k \in \mathbb{Z} \tag{19}$$

we rewrite (18) as

$$2\cos\frac{\pi\lambda}{3}\sinh(\lambda\theta) I_a(\theta) = \sinh(\lambda(\mathrm{i}\pi - \eta\theta)) I_a(2\mathrm{i}\pi - \theta) - \sinh(\lambda(\mathrm{i}\pi + \eta\theta)) I_a(2\mathrm{i}\pi + \theta), \tag{20}$$

always with $\eta = 1$. On the other hand, this equation coincides with (3) when $\eta = -1$. At $q = 1$ (i.e. $\lambda = 1/2$), however, the sign of η becomes immaterial and (20) exactly coincides with the equation satisfied by Ω_1.

The functional equation (20) has infinitely many solutions (a solution multiplied by a $2\mathrm{i}\pi$-periodic function of θ is a new solution) and it remains to be seen whether (9) with the identifications (19) and $\lambda = 1/2$ can yield the function Ω_1 relevant for the percolation problem.

From the known asymptotic behavior (see e.g. [19]) of the form factors (7) one deduces that $I_a(\theta)$ behaves as $\exp\left[\left(a - \frac{1}{2}\right)\theta\right]$ as $\theta \to +\infty$, a result which is not obvious from (9) but can be checked numerically. Comparing with (4) we see that I_a behaves asymptotically as Ω_1 provided we take $a = -1/6$, corresponding to the lower sign and $k = 0$ in (19) with $\lambda = 1/2$.

The value $\xi = 2\pi$ (i.e. $\lambda = 1/2$) falls in the repulsive regime of the sine-Gordon model in which the only singularity of the form factors (7) within the strip $\mathrm{Im}\,\theta \in (0, 2\pi)$ is the annihilation pole at $\theta = \mathrm{i}\pi$. Since $F_0(\theta)$ is free of poles in the strip, the annihilation pole must be carried by $A^a_\pm$, and then by I_a. Any other pole of I_a in the strip could not cancel simultaneously in A^a_+ and A^a_-, and then $I_a(\theta)|_{\xi=2\pi}$ possesses a single pole at $\theta = \mathrm{i}\pi$ in the strip $\mathrm{Im}\,\theta \in (0, 2\pi)$, exactly as it is the case for $\Omega_q(\theta)$ in the Potts model.

Summarizing, the functions $I_{-1/6}(\theta)|_{\xi=2\pi}$ and $\Omega_1(\theta)$ satisfy the same functional relations, and have the same asymptotic behavior and singularity structure; then we conclude that they coincide up to a normalization. Since we know that [19]

$$\mathrm{Res}_{\theta=\mathrm{i}\pi} F^a_{+-}(\theta) = \mathrm{i}(1 - \mathrm{e}^{-2\mathrm{i}\pi a})\langle \mathrm{e}^{\mathrm{i}a\beta\varphi}\rangle, \tag{21}$$

we read from (7), (8) and (17) that $I_a(\theta)$ has residue i on the pole. Knowing also that the percolative order parameter P (probability that a site belongs to an infinite cluster) is related to the Potts magnetization as[8]

$$P = \lim_{q\to 1}\frac{q}{q-1} M, \tag{22}$$

and recalling (1), we conclude that $\Omega_1(\theta)$ has residue $\mathrm{i}P$ on the pole, and therefore

$$\Omega_1(\theta) = P I_{-1/6}(\theta)|_{\xi=2\pi}. \tag{23}$$

[8] Relation (22) is written incorrectly in [3], see [8].

The values $\xi = 2\pi, a = -1/6$ fall in the range where (9) can be used to compute $\Omega_1(\theta)$ for real values of θ, which is sufficient for our purposes.

Near the percolation threshold the relations

$$S \simeq \Gamma^{\pm}|p-p_c|^{-\gamma}, \tag{24}$$

$$\xi \simeq f^{\pm}|p-p_c|^{-\nu}, \tag{25}$$

$$P \simeq B(p-p_c)^{\beta}, \tag{26}$$

$$\frac{\langle N_c\rangle}{N} \simeq A^{\pm}|p-p_c|^{2-\alpha} \tag{27}$$

define the critical amplitudes for the mean cluster size, the correlation length, the order parameter and the singular part of the mean cluster number per site, respectively; the superscripts $\pm$ refer to[9] $p \to p_c^{\mp}$. Below we consider both the second moment correlation length:

$$\xi_{2nd}^2 = \frac{1}{4}\frac{\int \mathrm{d}^2x|x|^2 g_c(x)}{\int \mathrm{d}^2x\, g_c(x)}, \tag{28}$$

and the true correlation length ξ_t defined by

$$g_c(x) \sim \mathrm{e}^{-|x|/\xi_t}, \qquad |x| \to \infty, \tag{29}$$

where $g_c(x)$ is the probability that x and the origin belong to the same finite cluster. It was shown in [3] that, in terms of the Potts kink mass m, ξ_t is $1/m$ at $p < p_c$ and $1/2m$ at $p > p_c$, and that

$$A^{\pm} = -\frac{1}{2\sqrt{3}\left(f_t^{+}\right)^2}. \tag{30}$$

Defining the amplitude combinations

$$R_{\xi}^{+} = [\alpha(1-\alpha)(2-\alpha)A^{+}]^{1/2} f^{+}, \qquad U = 4\frac{B^2\left(f_{2\mathrm{nd}}^{+}\right)^2}{\Gamma^{+}}, \tag{31}$$

which are universal due to the scaling relations $2-\alpha = 2\nu$ and $2\nu = 2\beta+\gamma$, (30) together with $\alpha = -2/3$ imply in particular

$$R_{\xi_t}^{+} = \left[\frac{40}{27\sqrt{3}}\right]^{1/2} = 0.9248\ldots, \tag{32}$$

a result recovered from a lattice computation in [21]. The result for $R_{\xi_{2\mathrm{nd}}}^{+}$ in the two-kink approximation was computed in [3] and compares quite well with the lattice estimate obtained from the combination of the data collected in table 1.

The result (23) allows us to complete the two-kink computation of the universal ratios involving the amplitudes $f_{2\mathrm{nd}}^{-}$, Γ^{-}, B. All the other necessary information was already given in [3] and here we only recall how the results for percolation follow from those of the Potts model when $q \to 1$.

Consider as an example the cluster size S. This is the limit of the Potts susceptibility divided by $q-1$, and the susceptibility is in turn the integral on the plane of the connected Potts spin–spin correlator. At $T < T_c$ the leading large distance contribution to this correlator is produced by a two-kink state and is multiplied by $q-1$ (the number of two-kink intermediate states in the low-temperature spectral sum). There are no other factors of $q-1$ since in this phase the

[9] We keep the notation of [3] where $\pm$ referred to the high/low-temperature Potts phases; we drop instead the tilde used there on percolation amplitudes.

Table 1. Lattice estimates of critical amplitudes for site percolation on triangular and square lattice. The superscripts a, b indicate [22, 23], respectively.

	Triangular	Square
A^+	-4.37^{a}	–
Γ^+	0.0720^{b}	0.102^{b}
B	0.780^{b}	0.910^{b}
$2f^+_{2\mathrm{nd}}$	0.520^{b}	0.520^{b}

Table 2. Universal amplitude ratios in two-dimensional percolation. The field theory results in the first two lines are exact; the others are obtained in the two-kink approximation. The superscripts a, b, c, d, e indicate [6, 7, 22–24], respectively.

	Field theory	Lattice
A^+/A^-	1	1^{a}
f_t^+/f_t^-	2	–
$f^+_{2\mathrm{nd}}/f_t^+$	1.001	–
$f^+_{2\mathrm{nd}}/f^-_{2\mathrm{nd}}$	3.73	$4.0 \pm 0.5^{\mathrm{c}}$
Γ^+/Γ^-	160.2	$162.5 \pm 2^{\mathrm{d}}$
U	2.22	$2.23 \pm 0.10^{\mathrm{e}}$
$R^+_{\xi_{2\mathrm{nd}}}$	0.926	$\approx 0.93^{\mathrm{a+b}}$

spin two-kink form factor $F_1^\sigma(\theta)$ is the product of $\Omega_q(\theta)$ times another function which is also finite at $q = 1$ (see [3]). At $T > T_c$ the spin–spin correlator coincides by duality with the low-temperature disorder–disorder correlator. The latter receives the leading contribution from a single one-kink state weighted by the squared disorder form factor $\left|F_K^\mu\right|^2 = M\left|F_1^\sigma(\infty)\right|$ [3]. As a consequence, due to (22), also the high-temperature Potts correlator vanishes as $q - 1$ in the percolation limit (the two-kink contribution behaves in the same way). Summarizing, the factors of $q - 1$ can be explicitly isolated and cancel in the computation of the percolative critical amplitudes for the cluster size. The same can be shown for the other amplitudes.

The field theoretical results for the complete list of *independent*[10] ratios involving the amplitudes (24)–(27) are summarized in table 2 together with the most accurate lattice estimates. As remarked above, the comparison confirms in particular the effectiveness of the two-particle approximated form factor results in integrable field theory.

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